JOURNAL OF APPROXIMATION THEORY 11, 343–349 (1974)

Approximation of an Entire Function

O. P. JUNEJA

Department of Mathematics, Indian Institute of Technology, Kanpur 208016, India

Communicated by Oved Shisha

1. INTRODUCTION

Let $f(x)$ be a real-valued continuous function defined on $[-1, 1]$, and let

$$E_n(f) \equiv \inf_{p \in \pi_n} \|f - p\|_{L^\infty[-1,1]}, \qquad n = 0, 1, 2,..., \tag{1}$$

be the minimum error in the Chebyshev approximation of $f(x)$ over the set π_n of real polynomials of degree at most n. Bernstein ([1], p. 118) proved that

$$\lim_{n\to\infty} E_n^{1/n}(f) = 0 \tag{2}$$

if, and only if, $f(x)$ is the restriction to $[-1, 1]$ of an entire function.

Let $f(z)$ be an entire function, and let

$$M(r) \equiv M_f(r) = \max_{|z|=r} |f(z)|;$$

then the order ρ and lower order λ of $f(z)$ are defined by ([2], p. 8)

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \log M(r)}{\log r} = \begin{matrix}\rho \\ \lambda\end{matrix} \qquad (0 \leqslant \lambda \leqslant \rho \leqslant \infty). \tag{3}$$

Now, for $f(z)$ entire, (2) does not give any clue as to the rate at which $E_n^{1/n}(f)$ tends to zero. Recently, Varga ([9], Theorem 1) has shown that this rate depends on the order of $f(z)$. In fact, he has proved that

$$\limsup_{n\to\infty} \frac{n \log n}{\log[1/E_n(f)]} = \rho, \tag{4}$$

where ρ is a nonnegative real number if, and only if, $f(x)$ is the restriction to $[-1, 1]$ of an entire function of order ρ.

However, if $f(z)$ is an entire function of infinite order, then (4) fails to give satisfactory information about the rate of decrease of $E_n^{1/n}(f)$. Reddy ([7], Theorem 1), making use of the concept of "index" of an entire function earlier introduced by Sato ([8], p. 412) extended the above result to functions of infinite order. Thus, if

$$\rho(q) = \limsup_{r\to\infty} \frac{\log^{[q]} M(r)}{\log r}, \qquad q \geqslant 2 \tag{5}$$

where $\log^{[0]} M(r) = M(r)$ and $\log^{[q]} M(r) = \log(\log^{[q-1]} M(r))$, then $f(z)$ is said to be of index q if $\rho(q-1) = \infty$ while $\rho(q) < \infty$. If $f(z)$ is of index q we shall call $\rho(q)$ the q-order of $f(z)$. Analogous to lower order, the concept of lower q-order can be introduced. Thus $f(z)$, an entire function of index q, is said to be of lower q-order $\lambda(q)$ if

$$\lambda(q) = \liminf_{r\to\infty} \frac{\log^{[q]} M(r)}{\log r}, \qquad q \geqslant 2. \tag{6}$$

Reddy ([7], Theorem 1) has extended (4) to functions of infinite order by showing that

$$\limsup_{n\to\infty} \frac{n \log^{[q-1]} n}{\log[1/E_n(f)]} = \sigma \tag{7}$$

satisfies $0 < \sigma < \infty$ if, and only if, $f(x)$ is the restriction to $[-1, 1]$ of an entire function of index q, with $\rho(q) = \sigma$.

However, the result corresponding to (7) does not always hold for the lower q-order. In fact, Reddy ([7], Theorems 2A and 2B) has further shown that if $f(x)$ is the restriction to $[-1, 1]$ of an entire function of index q, then its lower q-order $\lambda(q)$ satisfies

$$\liminf_{n\to\infty} \frac{n \log^{[q-1]} n}{\log[1/E_n(f)]} \leqslant \lambda(q) \tag{8}$$

and that the reverse inequality (and hence equality) holds in (8) if $E_{n-1}(f)/E_n(f)$ is a nondecreasing function of n for $n > n_0$. (9)

In the present paper, we obtain a result corresponding to (7) for the lower q-order $\lambda(q)$ which holds without the condition (9). We also give one more relation which depicts the influence of $\lambda(q)$ on the rate of decrease of $E_n(f)$.

2. Main Result

We first prove

THEOREM 1. *Let $f(x)$ be a real-valued continuous function which is the restriction to $[-1, 1]$ of an entire function $f(z)$ of index q ($\geqslant 2$). Then, $f(z)$ is of lower q-order $\lambda(q)$ if, and only if,*

$$\lambda(q) = \max_{\{n_h\}} \liminf_{h\to\infty} \frac{n_h \log^{[q-1]} n_{h-1}}{\log[1/E_{n_h}(f)]}, \tag{10}$$

where maximum is taken over all increasing sequences $\{n_h\}$ of natural numbers.

We require a few lemmas.

LEMMA 1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of index q ($\geqslant 2$) and lower q-order $\lambda(q)$ and let $\nu(r)$ denote the rank of the maximum term $\mu(r)$ for $|z| = r$, i.e., $\mu(r) = \max_{n\geqslant 0}\{|a_n| r^n\}$ and $\nu(r) = \max\{n \mid \mu(r) = |a_n| r^n\}$. Then*

$$\lambda(q) = \liminf_{r\to\infty} \frac{\log^{[q-1]} \nu(r)}{\log r} = \liminf_{r\to\infty} \frac{\log^{[q]} \mu(r)}{\log r}. \tag{11}$$

The lemma follows easily on the same lines as those of Whittaker ([10], Theorem 1) for $q = 2$, so we omit the proof.

LEMMA 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of index q ($\geqslant 2$) and lower q-order $\lambda(q)$ and let $\{n_k\}$ denote the range of the step function $\nu(r)$, then*

$$\lambda(q) = \liminf_{k\to\infty} \frac{\log^{[q-1]} n_k}{\log \rho(n_{k+1})} \tag{12}$$

where the $\rho(n_k)$ terms denote the jump points of $\nu(r)$.

Proof. It follows, from given data, that

$$\nu(r) = n_k \qquad \text{when} \qquad \rho(n_k) \leqslant r < \rho(n_{k+1})$$

and that

$$\rho(n_k) < \rho(n_k + 1) = \cdots = \rho(n_{k+1}).$$

Furthermore, if $n_k < n \leqslant n_{k+1}$, then $\rho(n) = \rho(n_{k+1})$ and so (11) gives

$$\lambda(q) = \liminf_{n\to\infty} \frac{\log^{[q-1]} n}{\log \rho(n)} \geqslant \liminf_{k\to\infty} \frac{\log^{[q-1]}(n_k + 1)}{\log \rho(n_{k+1})}$$

$$= \liminf_{k\to\infty} \frac{\log^{[q-1]}(n_k + 1)}{\log \rho(n_k + 1)} \geqslant \lambda(q),$$

which gives (12).

Remark. For $q = 2$, relation (12) is due to Gray and Shah ([3], Lemma 1).

LEMMA 3. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ be an entire function of index q ($\geqslant 2$) and lower q-order $\lambda(q)$ such that $\psi(k) \equiv |a_k/a_{k+1}|^{1/(n_{k+1}-n_k)}$ forms an increasing function of k for $k > k_0$; then*

$$\lambda(q) = \liminf_{k\to\infty} \frac{(n_{k+1} - n_k) \log^{[q-1]} n_k}{\log |a_k/a_{k+1}|}. \tag{13}$$

Proof. For $q = 2$, this result is due to Juneja and Kapoor [5]. We note that since $\psi(k)$ forms an increasing function of k for $k > k_0$, we have

$$\nu(r) = n_k \qquad \text{for} \quad \psi(k-1) \leqslant r < \psi(k),$$

so that, for sufficiently large k, $\rho(n_k) = \psi(k-1)$, $\rho(n_{k+1}) = \psi(k)$. Substituting the value of $\rho(n_{k+1})$ in (12) we get (13).

LEMMA 4. *Let $\{n_k\}$ be an increasing sequence of positive integers and let $\{a_n\}$ be a sequence of complex numbers such that $|a_{n_k}| < 1$ for $k > k_0$; then for $q \geqslant 2$,*

$$\liminf_{k\to\infty} \frac{n_k \log^{[q-1]} n_{k-1}}{\log |a_{n_k}|^{-1}} \geqslant \liminf_{k\to\infty} \frac{(n_k - n_{k-1}) \log^{[q-1]} n_{k-1}}{\log |a_{n_{k-1}}/a_{n_k}|}. \tag{14}$$

The lemma follows exactly along the same lines as those of Juneja ([4], Lemma 2) for $q = 2$, so we omit the proof.

Proof of Theorem 1. First, suppose that $f(x)$ has an analytic extension $f(z)$ which is an entire function of index q and lower q-order $\lambda(q)$. Following Bernstein's original proof ([6], p. 76) it follows that for each $n \geqslant 0$

$$E_n(f) \leqslant \frac{2B(r)}{r^n(r-1)} \qquad \text{for any} \quad r > 1 \tag{15}$$

where $B(r) = \max_{z\in\mathscr{C}_r} |f(z)|$, and $\mathscr{C}_r$ with $r > 1$ denotes the closed interior of the ellipse with foci ± 1, with half-major axis $(r^2+1)/2r$ and half-minor axis $(r^2-1)/2r$. The closed disks $D_1(r)$ and $D_2(r)$ bound the ellipse $\mathscr{C}_r$ in the sense that

$$D_1(r) \equiv \left\{z \,\middle|\, |z| \leqslant \frac{r^2-1}{2r}\right\} \subset \mathscr{C}_r \subset D_2(r) \equiv \left\{z \,\middle|\, |z| \leqslant \frac{r^2+1}{2r}\right\}.$$

From this inclusion, it follows that

$$M\left(\frac{r^2-1}{2r}\right) \leqslant B(r) \leqslant M\left(\frac{r^2+1}{2r}\right) \qquad \text{for all} \quad r > 1. \tag{16}$$

Consequently, (15) and (16) give for any sequence $\{n_h\}$ of positive integers that

$$M\left(\frac{r^2+1}{2r}\right) \geqslant E_{n_h}(f)\, r^{n_h} \qquad \text{for any } r > 3 \text{ and } h = 1, 2, \ldots. \tag{17}$$

Now, let

$$\liminf_{h\to\infty} \frac{n_h \log^{[q-1]} n_{h-1}}{\log(1/E_{n_h}(f))} = \alpha(\{n_h\}) \equiv \alpha.$$

Since $f(z)$ is an entire function, (2) gives $0 \leqslant \alpha \leqslant \infty$. First, let $0 < \alpha < \infty$; then for $\alpha > \epsilon > 0$,

$$E_{n_h}(f) > [\log^{[q-2]} n_{h-1}]^{-n_h/(\alpha-\epsilon)} \qquad \text{for} \quad h > h_0 = h_0(\epsilon).$$

Let $r_h = e(\log^{[q-2]} n_{h-1})^{1/(\alpha-\epsilon)}$ for $h = 1, 2, 3,\ldots$. If $r_h \leqslant r \leqslant r_{h+1}$, $h > h_0$, then (17) gives

$$\begin{aligned}\log M\left(\frac{r^2+1}{2r}\right) &\geqslant \log E_{n_h}(f) + n_h \log r \\ &\geqslant \log E_{n_h}(f) + n_h \log r_h \\ &> n_h \\ &= \exp^{[q-2]}\left(\frac{r_{h+1}}{e}\right)^{\alpha-\epsilon}.\end{aligned}$$

So,

$$\begin{aligned}\log^{[q]} M\left(\frac{r^2+1}{2r}\right) &> (\alpha-\epsilon)\log r_{h+1} - (\alpha-\epsilon) \\ &\geqslant (\alpha-\epsilon)\log r - (\alpha-\epsilon)\end{aligned}$$

or

$$\lambda(q) \equiv \liminf_{r\to\infty} \frac{\log^{[q]} M(r)}{\log r} \geqslant \alpha$$

which obviously holds when $\alpha = 0$. Since this inequality holds for every increasing sequence $\{n_h\}$ of positive integers, we have

$$\lambda(q) \geqslant \max_{\{n_h\}} \alpha(\{n_h\}) = \beta, \quad \text{for instance.} \tag{18}$$

Now, for each $n \geqslant 0$, there exists a unique $p_n(x) \in \pi_n$ such that

$$\| f - p_n \|_{L^\infty[-1,1]} = E_n(f), \qquad n = 0, 1, 2,\ldots .$$

Further, since $\| p_{n+1} - p_n \|_{L^\infty[-1,1]}$ is bounded above by $2E_n(f)$, we have by [6], p. 42:

$$| p_{n+1}(z) - p_n(z)| \leqslant 2E_n(f)\, r^{n+1} \text{ for all } z \in \mathscr{C}_r \text{ for any } r > 1. \tag{19}$$

Thus, we can write

$$f(z) = p_0(z) + \sum_{k=0}^{\infty} (p_{k+1}(z) - p_k(z)),$$

and this series converges uniformly in any bounded domain of the complex plane. So, (19) gives

$$| f(z)| \leqslant | p_0(z)| + 2 \sum_{k=0}^{\infty} E_k(f)\, r^{k+1} \qquad \text{for any} \quad z \in \mathscr{C}_r$$

and consequently, from the definition of $B(r)$

$$B(r) \leqslant A_0 + 2 \sum_{k=0}^{\infty} E_k(f)\, r^{k+1}.$$

So, (16) gives

$$M\left(\frac{r^2-1}{2r}\right) \leqslant A_0 + 2 \sum_{k=0}^{\infty} E_k(f)\, r^{k+1}. \tag{20}$$

Obviously, the function $g(z) = \sum_{k=0}^{\infty} E_k(f)\, z^{k+1}$ is an entire function. Let $\{n_k\}$ denote the range of $\nu(r)$ for this function. Consider the function $h(z) = \sum_{k=0}^{\infty} E_{n_k}(f)\, z^{n_k}$. It is easily seen that $h(z)$ is also an entire function and that $g(z)$ and $h(z)$ have the same maximum term for every z. It follows, from Lemma 1, that both have the same lower q-order. If we denote this by $\lambda_0(q)$, then since $h(z)$ satisfies the hypotheses of Lemma 3, we have

$$\begin{aligned} \lambda_0(q) &= \liminf_{k\to\infty} \frac{(n_k - n_{k-1}) \log^{[q-1]} n_{k-1}}{\log(E_{n_{k-1}}(f)/E_{n_k}(f))} \\ &\leqslant \liminf_{k\to\infty} \frac{n_k \log^{[q-1]} n_{k-1}}{\log(1/E_{n_k}(f))}, \qquad \text{by Lemma 4} \\ &\leqslant \max_{\{n_h\}} \liminf_{h\to\infty} \frac{n_h \log^{[q-1]} n_{h-1}}{\log(1/E_{n_h}(f))} = \beta. \end{aligned} \tag{21}$$

Thus (20) and (21) give

$$\begin{aligned} M\left(\frac{r^2-1}{2r}\right) &\leqslant A_0 + 2g(r) \\ &< 0(1) + 2 \exp^{[q-1]}(r^{\beta+\epsilon}) \quad \text{for a sequence } r_1, r_2, \ldots \to \infty. \end{aligned}$$

Hence, it follows that

$$\lambda(q) \leqslant \beta, \tag{22}$$

which shows that the lower q-order of $f(z)$ does not exceed β. Thus, if $f(z)$ is of lower q-order $\lambda(q)$, then (18) shows that $\beta \leqslant \lambda(q)$. If $\beta < \lambda(q)$, then the above arguments show that $f(z)$ would be of lower k-order less than β, a contradiction. Thus, we must have $\beta = \lambda(q)$. This completes the proof of the theorem.

Using Lemmas 3 and 4 and arguing as above the following theorem can also be proved easily.

THEOREM 2. *Let $f(x)$ be a real-valued continuous function which is the restriction to $[-1, 1]$ of an entire function $f(z)$ of index q. Then, $f(z)$ is of lower q-order $\lambda(q)$ if, and only if,*

$$\lambda(q) = \max_{\{n_h\}} \liminf_{h\to\infty} \frac{(n_h - n_{h-1}) \log^{[q-1]} n_{h-1}}{\log(E_{n_{h-1}}(f)/E_{n_h}(f))}, \tag{23}$$

where maximum is taken over all increasing sequences $\{n_h\}$ of natural numbers.

REFERENCES

1. S. BERNSTEIN, "Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable reille," Gauthier-Villars, Paris, 1926.
2. R. P. BOAS, JR., "Entire Functions," Academic Press, New York, 1954.
3. A. GRAY AND S. M. SHAH, Holomorphic functions with gap power series, *Math. Z.* **86** (1965), 375–394.
4. O. P. JUNEJA, On the coefficients of an entire series, *J. Analyse Math.* **24** (1971), 395–401.
5. O. P. JUNEJA AND G. P. KAPOOR, On the lower order of entire functions, *J. London Math. Soc.* (2) **5** (1972), 310–312.
6. G. G. LORENTZ, "Approximation of Functions," Holt, Rinchart and Winston, New York, 1966.
7. A. R. REDDY, Approximation of an entire function, *J. Approximation Theory* **3** (1970), 128–137.
8. D. SATO, On the rate of growth of entire functions of fast growth, *Bull. Amer. Math. Soc.* **69** (1963), 411–414.
9. R. S. VARGA, On an extension of a result of S. N. Bernstein, *J. Approximation Theory* **1** (1968), 176–179.
10. J. M. WHITTAKER, The lower order of integral functions, *J. London Math. Soc.* **8** (1933), 20–27.